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Measure of non-coaxiality

S. K. GHOSH

Department of Geological Sciences, Jadavpur University, Calcutta 700 032, India

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Abstract—The kinematical vorticity number (W_k) can be a measure of instantaneous non-coaxiality of geologic deformations. In order to determine the degree of non-coaxiality from finite deformation structures, an entity W_k' is defined. $W_k' = 2\Omega/D$, where Ω is the amount of rotation of a rigid equant grain embedded in a ductile matrix, $D = [2(\bar{\varepsilon}_1^2 + \bar{\varepsilon}_2^2 + \bar{\varepsilon}_3^2)]^{1/2}$ and $\bar{\varepsilon}_i$ are the principal natural strains. For different values of W_k the progressive variation of 2Ω with D is plotted graphically. The gradient of each curve at the origin is equal to W_k. The gradient of the curves and the values of W_k' increase with progressive deformation. If Ω and D are independently determined from the rock fabric, then the plots of the data on the $2\Omega/D$ diagram will enable us to identify a curve for a particular W_k. The analysis is carried out for plane equivoluminal strain and the method is applicable for the case in which the instantaneous character of deformation remains unchanged in progressive deformation.

INTRODUCTION

THERE have been some attempts to find a suitable measure for the non-coaxiality or the rotationality of deformation of rocks with respect to an internal frame of reference (Elliott 1972, Means *et al.* 1980). Such a measure is not concerned with the magnitude of non-coaxial deformation nor merely with faster or slower rates of deformation; any two simple shear deformations, with a fast and a slow rate, are equally non-coaxial. As Truesdell (1954) pointed out, "the measure of rotation should indicate not the relative speeds but the rotational quality or degree". This measure then could be a basis to separate coaxial and non-coaxial deformations as well as to differentiate different classes of non-coaxial deformations.

CLASSIFICATION OF DEFORMATIONS ON THE BASIS OF KINEMATICAL VORTICITY NUMBER

As suggested by Means *et al.* (1980), for steady progressive deformation (i.e. a deformation the instantaneous character of which remains constant during a certain interval of time) the kinematical vorticity number of Truesdell (1954, p. 107) can be a measure of non-coaxiality. The kinematical vorticity number is a dimensionless invariant which measures the degree of rotationality of deformation with respect to an internal reference frame and is defined by

$$W_k = \frac{w}{[2(\dot{\varepsilon}_1^2 + \dot{\varepsilon}_2^2 + \dot{\varepsilon}_3^2)]^{1/2}}, \tag{1}$$

where $\dot{\varepsilon}_i$ are the linear strain rates and w is the magnitude of the vorticity vector.

Since we are interested essentially in the physical expression of vorticity, it is worth noting that, in a plane perpendicular to the direction of the vorticity vector, the magnitude of vorticity is equal to twice the magnitude of the local angular velocity, which again is equal to: (1) the average of the rates of rotation of material lines of all directions in the plane, (2) the average rate of rotation of any two material lines perpendicular to each other and lying on the plane, (3) the rate of rotation of a rigid spherical inclusion and (4) the rate of rotation of a rigid ellipsoidal inclusion whose principal axes are aligned along the principal axes of the strain rate ellipsoid. For our purpose the third item among these is the most important; the magnitude of vorticity is twice the rotation rate of a rigid equant grain, such as a porphyroblast of garnet, embedded in a much softer matrix.

Means *et al.* (1980) have shown that, depending on the value of the kinematical vorticity number, steady progressive deformations may be classified into the following categories: (1) coaxial deformation history with $W_k = 0$, (2) non-pulsating deformation history with $0 < W_k < 1$, (3) progressive simple shear with $W_k = 1$, (4) pulsating deformation history (Ramberg 1975) with $1 < W_k < \infty$ and (5) rigid rotation with $W_k = \infty$. These types are also characterized respectively by five different types of particle paths: rectangular hyperbola, non-rectangular hyperbola, straight line, ellipse and circle.

ESTIMATION OF W_k FROM ROCK FABRIC

The following discussion is restricted to plane strain; the analysis can be easily extended to the cases in which the axis of rotation is either a direction of shortening or a direction of extension. Consider a plane deformation of the type relevant for our purpose:

$$\begin{aligned} u &= a_{11}x + a_{12}y \\ v &= a_{21}x + a_{22}y, \end{aligned} \tag{2}$$

where u and v are the velocities along x and y axes, respectively. The magnitude of vorticity is then: $w = a_{12} - a_{21}$ and the kinematical vorticity number is

$$W_k = \frac{a_{12} - a_{21}}{\sqrt{2}\sqrt{[a_{11}^2 + a_{22}^2 + \frac{1}{2}(a_{12} + a_{21})^2]}}. \tag{3}$$

For isochoric deformation $a_{11} + a_{22} = 0$ and

$$W_k = \frac{a_{12} - a_{21}}{\sqrt{[(a_{12} + a_{21})^2 - 4a_{11}a_{22}]}}. \tag{4}$$

Thus for isochoric deformation:

$$\begin{aligned} a_{12}a_{21} - a_{11}a_{22} &= 0, \quad \text{when } W_k = 1,\\ a_{12}a_{21} - a_{11}a_{22} &> 0, \quad \text{when } W_k < 1,\\ a_{12}a_{21} - a_{11}a_{22} &< 0, \quad \text{when } W_k > 1. \end{aligned}$$

Alternatively, we may choose the coordinate axes along the principal directions and represent the deformation by a superposition of a rigid rotation on a pure strain. Then, for plane isochoric deformations, $\omega = \dot{\varepsilon}_1$ when $W_k = 1$, $\omega < \dot{\varepsilon}_1$ when $W_k < 1$ and $\omega > \dot{\varepsilon}_1$ when $W_k > 1$. Here ω is the magnitude of local angular velocity and $\dot{\varepsilon}_1$ is the greatest principal strain rate.

Instantaneous plane strain may also be represented by a combination of pure shear and simple shear (Ramberg 1975) with the simple shear plane at an angle of θ to a principal axis of pure shear. From (3) the kinematical vorticity number will then be

$$W_k = \frac{1}{[\cos^2 2\theta + (2s_r - \sin 2\theta)^2]^{1/2}}, \tag{5}$$

where s_r is the ratio $\dot{\varepsilon}_x/\dot{\gamma}$ between the rates of pure shear and simple shear. If $s_r \neq 0$, we have

$$\begin{aligned} s_r - \sin 2\theta &= 0, \quad \text{when } W_k = 1,\\ s_r - \sin 2\theta &> 0, \quad \text{when } W_k < 1,\\ s_r - \sin 2\theta &< 0, \quad \text{when } W_k > 1. \end{aligned}$$

Equation (5) shows that there may be an infinite number of combinations of s_r and θ for which we obtain an identical shape of the particle path. Thus, a straight line particle path is obtained for any combination of s_r and θ which satisfies the equation $s_r - \sin 2\theta = 0$. Again, for example, each of the combinations ($s_r = 0.75$, $\theta = 45°$), ($s_r = 0.25$, $\theta = 45°$), ($s_r = 0.727$, $\theta = 50°$) and ($s_r = 0.432$, $\theta = 30°$) will give the same shape of a hyperbolic particle path with $W_k = 0.5$. In this sense this method of combining pure shear and simple shear is somewhat arbitrary. Nevertheless, Ramberg's method of combining pure strain and simple shear is important because it gives us a simple physical representation of a complex type of deformation in which the character of instantaneous deformation remains unchanged in time. Moreover, once we fix the value of W_k, it is immaterial what values of s_r and θ are chosen, as long as the values satisfy equation (5).

The kinematical vorticity number measures the non-coaxiality of instantaneous deformation. Even if we assume that in an area the nature of instantaneous deformation did not change in course of time, the resulting fabric can give us only total strains and total rotations. The problem is thus to find a method to determine the degree of instantaneous non-coaxiality from finite deformation structures under the assumption that the nature of instantaneous deformation did not change in course of time.

Let us define an entity

$$W'_k = \frac{2\Omega}{[2(\bar{\varepsilon}_1^2 + \bar{\varepsilon}_2^2 + \bar{\varepsilon}_3^2)]^{1/2}}, \tag{6}$$

where $\bar{\varepsilon}_i$ are the natural or logarithmic principal strains and Ω is the amount of rotation of a rigid spherical inclusion. If the deformation is a progressive simple shear, $\Omega = \gamma/2$, where γ is the amount of simple shear. If we choose to represent a steady progressive deformation by combined pure strain and simple shear, Ω will again be equal to $\gamma/2$ (Ghosh & Ramberg 1976). Ω is thus measurable wherever the rock fabric enables us to determine the amount of rotation of an equant porphyroblast or the amount of lateral displacement of beds (from displacement of pre-existing transverse veins) or the amount of wall-parallel shear strain in ductile shear zones (Ramsay 1980). W'_k can be determined if the finite longitudinal strains can also be measured from natural strain gauges.

Let

$$D = [2(\bar{\varepsilon}_1^2 + \bar{\varepsilon}_2^2 + \bar{\varepsilon}_3^2)]^{1/2}.$$

Then, for plane isochoric deformation

$$D = 2\bar{\varepsilon}_1.$$

(1) For progressive simple shear, with $W_k = 1$, the natural strain along the X-axis is

$$\bar{\varepsilon}_1 = \sinh^{-1}(\gamma/2) \tag{7}$$

(Jaeger, 1962, p. 69). Thus, for isochoric deformation

$$D = 2\sinh^{-1}(\gamma/2) \tag{8}$$

and

$$W'_k = \frac{\gamma}{2\sinh^{-1}(\gamma/2)}.$$

When γ is very small $W'_k \approx 1$ and $W'_k \approx W_k$. With progressive deformation W'_k increases and may deviate considerably from W_k.

(2) For hyperbolic particle paths, with $W_k < 1$, it is convenient to put $\theta = 0$ in eqn (5) and obtain the relation

$$s_r = \frac{1}{2}\left(\frac{1}{W_k^2} - 1\right)^{1/2}. \tag{9}$$

For any particular value of W_k, s_r is calculated from eqn (9) and D is determined from the relation

$$D = 2\ln R_1, \tag{10}$$

where R_1, the semi-major axis of the strain ellipse, is given by the expression

$$R_1 = \frac{1}{2}\left[\left\{4 + \left(4 + \frac{1}{s_r^2}\right)\sinh^2(\gamma s_r)\right\}^{1/2} + \left\{\left(4 + \frac{1}{s_r^2}\right)\sinh^2(\gamma s_r)\right\}^{1/2}\right]. \tag{11}$$

Hence

$$W'_k = \frac{\gamma}{2\ln R_1}. \tag{12}$$

(3) For elliptical particle paths, with $W_k > 1, \theta \neq 0$. It is convenient to put $\theta = 45°$ in eqn (5). Then

$$s_r = \frac{1}{2}\left(1 \pm \frac{1}{W_k}\right). \tag{13}$$

For a fixed value of W_k, the shapes of the particle paths will be identical for both the values of s_r given by (13). With any one of these values R_1 can be determined from eqns (72) and (83) of Ramberg (1975). With $\theta = 45°$,

$$R_1 = \frac{1}{(N - T)^{1/2}}$$

where

$$N = 1 + \sin^2 (\gamma L)\left[\frac{P^2}{L^2} + \frac{1}{4L^2} - 1\right],$$

$$T = \left[\frac{P^2}{L^2}\sin^2 (2\gamma L) + \frac{P^2}{L^4}\sin^4 (\gamma L)\right]^{1/2},$$

$$P = s_r - \tfrac{1}{2},$$

$$L = \sqrt{[s_r(1 - s_r)]}.$$

D and W_k' are then calculated from eqns (10) and (12).

For different values of W_k, the progressive variation of 2Ω with D is shown in Fig. 1. In this figure the gradient of each curve at the origin is equal to W_k. The gradient of the curves and the value of W_k' increase with progressive deformation. Hence W_k' itself does not characterize the nature of instantaneous deformation. The graphs, however, show that, for relatively small values of W_k, there is very little change in W_k' even up to very large values of D. Hence when W_k is rather small, say less than 0.5, $W_k' \approx W_k$.

For $W_k > 1$, we enter into the field of pulsating strain histories and the 2Ω–D curves become periodic. Only one period of each curve is shown in Fig. 1. For such a strain history, with an elliptical particle path in a section perpendicular to the rotation axis, the strain ellipse pulsates; the maximum elongation is reached when

$$2\Omega = \frac{\pi}{\sqrt{[1 - (1/W_k)^2]}}. \tag{14}$$

At this value of 2Ω, the curve becomes parallel to the 2Ω axis. With further deformation the elongation decreases. The strain ellipse regains its circular shape when

$$2\Omega = \frac{2\pi}{\sqrt{[1 - (1/W_k)^2]}}. \tag{15}$$

This is the period of the 2Ω–D curves for pulsating strain histories and is equal to half the period of a particle to make a complete traverse along its elliptical path. The period of the 2Ω–D curves as given by (15) decreases with increasing W_k and reaches the limiting value of 2π when W_k tends to infinity and when the elliptical particle path degenerates to a circular path. We can see from Fig. 1 that for $W_k = 5$ we are very close to this limiting value.

To determine the kinematical vorticity number the general procedure is to determine independently Ω and D from the rock fabric and to plot the data on a 2Ω–D diagram (Fig. 1). If the curves are drawn for close-spaced values of W_k the point will plot on or very close to one of these curves. The identification of the curve will enable us to determine the kinematical vorticity number with sufficient accuracy.

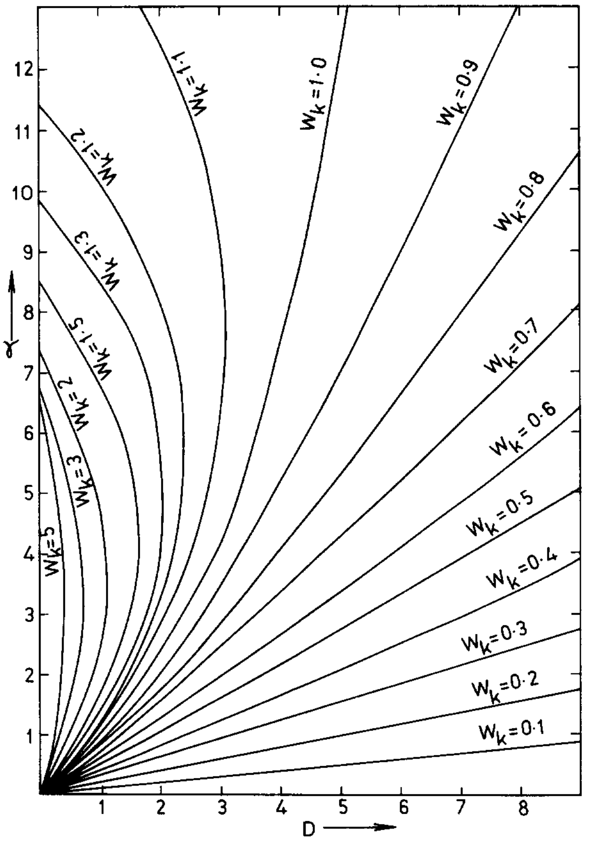

Fig. 1. The variation of 2Ω or γ with D in progressive deformation. The gradient of each curve at the origin is equal to W_k. Note that 2Ω is given in radians.

This analysis is made under the assumption that the value of W_k does not change with progressive deformation. As yet there are no field criteria to decide whether or not W_k remains constant in any given natural deformation. Nevertheless the plotting of the deformation data on a 2Ω–D diagram can be informative. In an area where the intensity of deformation varies spatially, the patterns of the 2Ω–D plots may tell us whether W_k was constant or not. The patterns of the plots may also indicate whether or not the deformation deviates significantly from the model of simple shear and whether pulsating strain histories, with $W_k > 1$, are geologically realistic.

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